Eur. Phys. J. B **9**, 533–543 (1999)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

Clusters in an assembly of globally coupled bistable oscillators

V.I. Nekorkin[1,2], M.L. Voronin[1], and M.G. Velarde[2,a]

[1] Nizhny Novgorod State University, 23 Gagarin Ave., 603600 Nizhny Novgorod, Russia
[2] Instituto Pluridisciplinar, Universidad Complutense, Paseo Juan XXIII, N° 1, Madrid 28 040, Spain

Received 12 June 1998 and Received in final form 30 November 1998

Abstract. We study the dynamics of an assembly of globally coupled *bistable* elements. We show that bistability of elements results in some new features of clustering in the assembly when there is *global* coupling. We provide conditions for the existence of stable amplitude-phase clusters and splay–phase states.

PACS. 05.45.-a Nonlinear dynamics and nonlinear dynamical systems

1 Introduction

Large assemblies of identical or almost identical interacting nonlinear elements play a significant role in understanding the dynamical behaviour of many systems which are studied in various fields of science [1–10]. In a wide class of such systems an important place is occupied by assemblies of *globally* coupled nonlinear elements. In such assemblies each element is coupled with equal strength to all others. Examples of globally coupled assemblies are oscillatory neuronal systems [3,7,8,10,11], arrays of Josephson junctions [12,13], some laser and electronic systems [14], *etc.* The dynamics of assemblies with *global* coupling is extremely varied and can be rather complex [15–20]. One interesting feature of such systems is the possibility of formation of subgroups of elements *i.e.* clustering [21–28], with specific properties different for each subgroup. In the case of weak coupling between elements a useful approximation to study the phenomenon of clustering is the so-called phase model [29]. This approach is based on the assumption that the amplitudes of the limit cycles, existing in each element of the assembly in the absence of coupling, do not noticeably change with weak enough coupling. Accordingly, amplitude equations are disregarded and we can focus in studying only phase equations to describe the dynamics of the assembly and to investigate phase clustering. But the phase model becomes inapplicable for assemblies with elements possessing *bistable* properties like two stable attractors. Bistability is a significant property of many nonlinear systems. For example, the bistable behaviour of neurons is used as a basic ingredient for some neural networks [3,7,8]. In the simplest approximation an individual neuron can be in two states: the state of the rest and the regime of periodic, limit cycle oscillations. For assemblies of *bistable* units we must, simultaneously, consider the equations for both amplitudes and phases. The bistability of a single element can drastically influence the collective behaviour of the assembly and lead to effects not observed in arrays of elements with a single attractor. Here we investigate the dynamics of an assembly of globally coupled *bistable* oscillators.

The paper is organized as follows. First, we introduce the model problem to be studied. In Section 2 we describe homogeneous oscillations. In section 3 we study in-phase motions in a *gradient* system. In our system, stable amplitude oscillations are possible. In Section 4 we consider amplitude-phase clusters. We find that introducing suitable parameters, α, γ, phase differences appear between clusters. In Section 5 we study the so-called "splay-phase" states. The conditions of their existence and stability are analytically obtained. In Section 6 we discuss new features of the expected irregular behaviour in the assembly of *bistable* oscillators. Finally, in the Conclusion we summarize the results obtained and hence we discuss the salient features of the influence of bistability on the dynamics of the system.

We consider an assembly of N globally coupled identical *bistable* oscillators evolving in time according to

$$\dot{W}_j = -W_j\left[f\big(|W_j|\big) - i\Big(\omega + \alpha g\big(|W_j|\big)\Big)\right] + (\beta + i\gamma)\left(\overline{W} - W_j\right), \quad j = 1, 2, \ldots, N, \tag{1}$$

where W_j is a complex variable, $f(|W|) = 2a|W|^4 - a|W|^2 + 1$, $g(|W|) = |W|^2 - 2|W|^4$, and

$$\overline{W} = \frac{1}{N}\sum_{k=1}^{N} W_k. \tag{2}$$

The functions $f(|W|)$, $g(|W|)$ and the parameters ω, a, α characterize the individual dynamics of the element in the assembly. ω, a have positive values and $\alpha \le 0$. When

[a] e-mail: mvelarde@eucmax.sim.ucm.es

$\alpha = 0$ the motions of a single oscillator are isochronous, and when $\alpha < 0$ they are nonisochronous. For $a > 8$ the stable limit cycle and the stable steady state coexist with basins of attraction separated by an unstable limit cycle. The parameters β and γ characterize the strength of the coupling between the oscillators, and have positive values. This interaction operates only through the mean field $\overline{W}$. We assume that the assembly size, N, is suitably large. An alternative description of (1) is obtained using "polar" coordinates, $z_j = W_j \mathrm{e}^{i(\gamma-\omega)t}$. Then (1) becomes

$$\dot{z}_j = -z_j \left[h(|z_j|) - i\alpha g(|z_j|)\right] + (\beta + i\gamma)\bar{z}, \quad j = 1, 2, \ldots, N, \tag{3}$$

where $h(|z|) \equiv f(|z|) + \beta$, $\bar{z} = \dfrac{1}{N}\sum_{k=1}^{N} z_k$.

2 Homogeneous oscillations

Homogeneous oscillations of the assembly (1) correspond to the solutions of the system (3), independent of j. The system (3) has three homogeneous solutions

$$z_j(t) = r^0 \mathrm{e}^{i\left(g(r^0)t+\varphi^0\right)}, j = 1, 2, \ldots, N, \tag{4}$$

with

$$r^0 = \begin{cases} 0, \\ r^{(1)} \equiv \dfrac{1}{2}\sqrt{1-\sqrt{1-8/a}}, \; \varphi^0 = \text{const.} \\ r^{(2)} \equiv \dfrac{1}{2}\sqrt{1+\sqrt{1-8/a}}. \end{cases}$$

Linearizing the system (3), around each solution (4), we obtain for perturbations, $\xi_j \in \mathrm{C}$, the following equations for disturbances upon the trivial solution

$$\dot{\xi}_j = -\xi_j + (\beta + i\gamma)\left(\frac{1}{N}\sum_{k=1}^{N}\xi_k - \xi_j\right) \tag{5}$$

and for disturbances around the other homogeneous oscillations

$$\dot{\xi}_j = -H(r^0)\,(a + i\alpha)\left(\xi_j + \xi_j^*\right) + (\beta + i\gamma)\left(\frac{1}{N}\sum_{k=1}^{N}\xi_k - \xi_j\right). \tag{6}$$

Here and below, starred quantities denote complex conjugation, and $H(r) = r^2(4r^2 - 1)$. The matrices associated with systems (5, 6) are circulant, and their eigenvalues, which are the Lyapunov exponents of solutions (4), can be easily found. To do this, the matrices are reduced to the block-diagonal form [30]. We carried out such analysis and obtained the following results. The trivial solution $z_j = 0$ has the Lyapunov exponents

$$\lambda_1 = \lambda_2 = -1, \; \lambda_{2+s} = -(\beta + 1) \pm i\gamma, \quad s = 1, 2, \ldots, N-2 \tag{7}$$

and the nontrivial solutions have $\lambda_1 = 0$, $\lambda_2 = -2aH(r^0)$ and $(N-1)$ pairs of exponents, which are the roots of equation

$$\lambda^2 + 2(aH(r^0) + \beta)\lambda + \beta^2 + \gamma^2 + 2H(r^0)\,(a\beta + \alpha\gamma) = 0. \tag{8}$$

The analysis of the distribution of eigenvalues in the complex plane shows that the trivial solution of the assembly is stable, that homogeneous oscillations with amplitudes $r^{(1)}$ are unstable while those with $r^{(2)}$ are stable in the region defined by the inequality

$$\beta^2 + \gamma^2 + 2H(r^{(2)})\,(a\beta + \alpha\gamma) > 0. \tag{9}$$

Hence for some initial conditions all the oscillators of the assembly (1) can be at rest, and the others may exhibit homogeneous periodic oscillations.

3 Amplitude clusters

Let us consider the collective dynamics of the assembly (1), consisting of isochronous oscillators, ($\alpha = 0$), globally coupled with $\beta(\gamma = 0)$ now taken real. For $\alpha = \gamma = 0$ the system (3) is *gradient* as

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = -\frac{\partial U}{\partial z_j^*}, \tag{10}$$

with

$$U = \frac{1}{2}\sum_{j=1}^{N}\left\{G\left(|z_j|^2\right) + \frac{\beta}{N}\sum_{k=1}^{N}|z_k - z_j|^2\right\},$$

$$G\left(|z|^2\right) = 2|z|^2\left[1 - \frac{a}{2}|z|^2 + \frac{2a}{3}|z|^4\right].$$

Consequently, the system (3) has steady states only and for any initial conditions all trajectories tend to one of them. Hence amplitudes and phases of stable oscillations of the assembly (1) correspond to stable fixed points of (3). The simplest of these stationary points correspond to homogeneous oscillations.

3.1 Existence of in-phase motions

It follows from (3) that in the phase space of the system there exists a manifold of *in-phase* motions $\mathrm{S} = \left\{\varphi_j = \varphi^0 = \text{const}, \; j = 1, 2, \ldots, N\right\}$. On the manifold S the equations describing the dynamics of amplitudes have the form

$$\dot{r}_j = -F(r_j) + \frac{\beta}{N}\sum_{k=1}^{N}(r_k - r_j), \tag{11}$$

where $z_j = r_j \mathrm{e}^{i\varphi^0}, F(r) = 2ar^5 - ar^3 + r$. The coordinates of the steady states of the system (11) determine amplitudes of *in-phase* oscillations of the assembly (1).

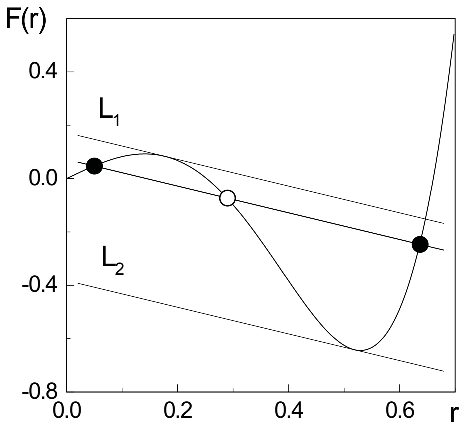


Fig. 1. Geometrical interpretation of conditions (12) and (13).

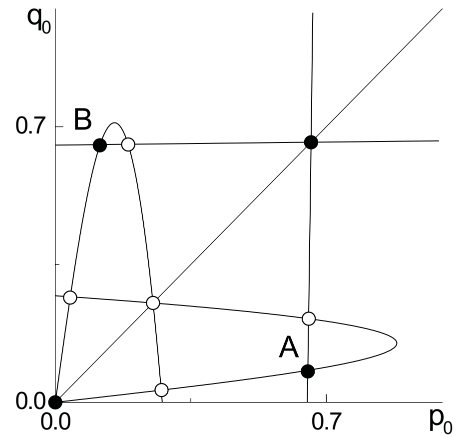


Fig. 2. Graphic solutions of the system (17).

Let us now find conditions of existence of inhomogeneous states of the system (11). We obtain from (11) that for each $i = 1, 2, ..., N-1$ the coordinates of the steady states obey the following relationship

$$F(r_{i+1}) - F(r_i) = -\beta(r_{i+1} - r_i). \tag{12}$$

On the other hand, according to Lagrange's theorem there exist $\rho_i \in (r_i, r_{i+1})$ such that

$$F(r_{i+1}) - F(r_i) = F'(\rho_i)(r_{i+1} - r_i). \tag{13}$$

Comparing (12) and (13), we conclude that

$$F'(\rho_i) = -\beta. \tag{14}$$

Hence, taking into account the form of $F(r)$, we see that condition (14) is satisfied in the parameter region delineated by the inequalities

$$\beta + 1 < 9a/40, \quad a > 8. \tag{15}$$

For the parameter values from this region on the curve $F(r)$ for $r \geq 0$ there are two points where the angular coefficients of the tangents are equal to $-\beta$ (Fig. 1, lines L_1 and L_2). From (13, 14) follows that the coordinates of the steady states of the system (11) must be abscissas of the points of intersection of the curve $F(r)$ with the secant, which is parallel to the tangents L_1, L_2 (Fig. 1). Consequently, the coordinates of the steady states form sets $\{r_j^0\}$, $j = 1, 2, ..., N$, whose elements have either two or three different digits. As shown below only in-phase oscillations, corresponding to the stationary points whose coordinates satisfy the condition $F'(r) + \beta > 0$, can be stable. Abscissas of the "middle" point in Figure 1 obviously violate this condition. Thus we turn to consider the steady states corresponding to the "extreme" points in Figure 1. Thus we look for sets $\{r_j^0\}$, consisting of two different positive numbers, denoted by p_0 and q_0. Since the system (11) is symmetric under permutation of the N indices, one can assume, without loss of generality, that

$$r_j = \begin{cases} p_0, & j = 1, 2, \ldots, n, \\ q_0, & j = n+1, \ldots, N. \end{cases} \tag{16}$$

From (11) we obtain the system of equations for p_0 and q_0 :

$$\begin{cases} q_0 = p_0 + \dfrac{N}{(N-n)\beta} F(p_0), \\ p_0 = \ q_0 + \dfrac{N}{n\,\beta} F(q_0). \end{cases} \tag{17}$$

Note that outside the region defined by (15), *i.e.* above the curve K in Figure 3, the system (17) can have only solutions that satisfy the condition $p_0 = q_0$ and correspond to homogeneous oscillations (as shown in Sect. 2). The largest number of real solutions of the system (17), such that $p_0 \neq q_0$, is six (Fig. 2). But since the system (17) is invariant under the transformations $\{n \to N-n,\ N-n \to n,\ q_0 \to p_0,\ p_0 \to q_0\}$ the solutions, satisfying the condition $p_0 > q_0$, coincide with the corresponding solutions with $p_0 < q_0$, which are numbered in a different way. Therefore, for example, we can restrict consideration to just three of them with $p_0 > q_0$. We show below that stable inhomogeneous in-phase oscillations have amplitude distributions $\{r_j^0\}$ of values p_{A} and q_{A} only, where p_{A}, q_{A} are the coordinates of the point A in Figure 2. For $\beta \ll 1$, using regular perturbation theory, one can find that

$$\begin{aligned} p_{\mathrm{A}} &= \beta\frac{(N-n)}{N} r^{(2)} + O(\beta^2), \\ q_{\mathrm{A}} &= r^{(2)} - \beta\frac{n}{2Nr^{(2)}\sqrt{a(a-8)}} + O(\beta^2), \end{aligned} \tag{18}$$

where $r^{(2)}$ is the amplitude of one of the homogeneous solutions, defined in formula (4).

Thus in the assembly described by system (1) there exist both homogeneous and inhomogeneous in-phase oscillations.

3.2 Stability of the inhomogeneous in-phase oscillations

Let $\{r_j = r_j^0,\ \varphi_j = \varphi_j^0\}$ be the solution of the system (3), providing inhomogeneous in-phase oscillations of the assembly. Linearization of the system (3) around this solution gives the following equations for perturbations, $\xi_j = r_j - r_j^0$ and $\eta_j = \varphi_j - \varphi^0$:

$$\begin{cases} \dot{\xi}_j = -\alpha_p \xi_j + d\sum_{k=1}^{N} \xi_k, \quad j = 1, 2, \dots, n, \\ \\ \dot{\xi}_j = -\alpha_q \xi_j + d\sum_{k=1}^{N} \xi_k, \quad j = n+1, \dots, N, \end{cases} \tag{19}$$

and

$$\begin{cases} \dot{\eta}_j = -d\sigma_1 \eta_j + d\sum_{i=1}^{n} \eta_i + d\frac{q_0}{p_0} \sum_{i=n+1}^{N} \eta_i, \\ j = 1, 2, \dots, n, \\ \dot{\eta}_j = -d\sigma_2 \eta_j + d\frac{p_0}{q_0}\sum_{i=1}^{n} \eta_i + d\sum_{i=n+1}^{N} \eta_i, j = n+1, \dots, N, \end{cases} \tag{20}$$

with

$$\alpha_p \equiv \beta + F'(p_0), \quad \alpha_q \equiv \beta + F'(q_0),$$
$$d \equiv \frac{\beta}{N}, \quad \sigma_1 \equiv n + \frac{q_0}{p_0}(N-n), \quad \sigma_2 \equiv N - n + \frac{p_0}{q_0}n.$$

It follows from (19, 20) that amplitude and phase disturbances evolve in an independent way.

Consider first the system (19) describing the evolution of amplitude perturbations. Let us introduce "difference" variables

$$\xi_{i+1} - \xi_i = u_i, \qquad i = 1, 2, \dots, n-1,$$
$$\xi_{i+1} - \xi_i = v_{i-n}, \qquad i = n+1, \dots, N-1,$$

and hence the new variables are governed by the system

$$\begin{cases} \dot{u}_i = -\alpha_p u_i, \; i = 1, 2, \dots, n-1, \\ \dot{v}_k = -\alpha_q v_k, \; k = 1, 2, \dots, N-n-1. \end{cases} \tag{21}$$

If $\alpha_p > 0$ and $\alpha_q > 0$, in the phase space of (19) there exists a stable manifold

$$M = \{\; \xi_1 = \xi_2 = \dots = \xi_n = u(t),$$
$$\xi_{n+1} = \xi_{n+2} = \dots = \xi_N = v(t) \;\}.$$

On the manifold M the evolution obeys the 2nd order differential system

$$\begin{cases} \dot{u} = (nd - \alpha_p)u + d(N-n)v, \\ \dot{v} = dnu + \big(d(N-n) - \alpha_q\big)v. \end{cases} \tag{22}$$

Consequently, if the manifold M and if the trivial solution of the system (22) are stable, all perturbations decay, $\xi_j \to 0$. This is the case in the parameter region defined by the inequalities

$$\beta + F'(p_0) > 0, \quad \beta + F'(q_0) > 0, \quad \beta + F'(p_0) + F'(q_0) > 0,$$
$$\beta \left\{ \frac{n}{N} F'(p_0) + \frac{(N-n)}{n} F'(q_0) \right\} + F'(p_0)F'(q_0) > 0. \tag{23}$$

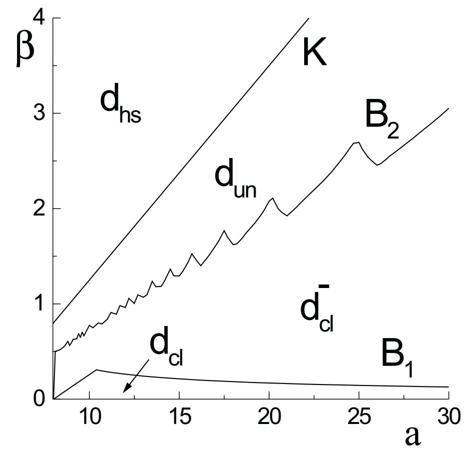


Fig. 3. Regions of existence and stability of homogeneous and inhomogeneous oscillations of the assembly ($N = 30$). The region $d_{\rm cl}$ defines the parameter values, for which $2^N - 2$ inhomogeneous states are stable. The region $d_{\rm cl}^-$ is where the amplitude-phase clusters become unstable. The line $\rm B_2$ bounds the region of stability of the inhomogeneous states. Above the curve K only homogeneous states can exist.

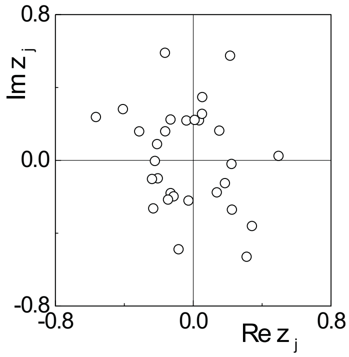
In a similar way one can show that one of the eigenvalues of (20) vanishes and all others are negative for all possible values of the parameters β and a.

Thus the inequalities (23) separate the region, wherein inhomogeneous steady states of system (3) and, consequently, inhomogeneous oscillations of the assembly are stable. Using (23), it can be shown that all inhomogeneous oscillations with amplitude distribution of values different form $p_{\rm A}$ and $q_{\rm A}$ are unstable while those having these values can be either stable or unstable.

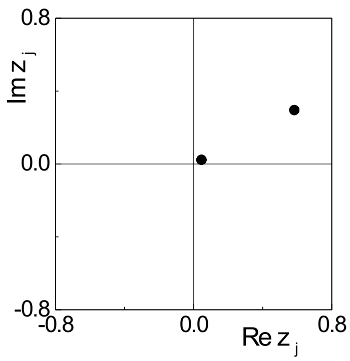
3.3 Formation of amplitude clusters

From equations (18) it follows that for $\beta \ll 1$, $F'(p_0) > 0$, $F'(q_0) > 0$, which ensures fulfillment of the stability conditions (23) for all n. Thus in this case there are $2^N - 2$ stable inhomogeneous states of the system (3). Solving, numerically, the system (17) and inequalities (23) we obtain, that all these solutions exist and are stable not only for $\beta \ll 1$, but also in some region $d_{\rm cl}$ (Fig. 3). Each of such steady states has its own set $\{r_j^0\}$, $j = 1, 2, ..., N$, of values $p_{\rm A}$ and $q_{\rm A}$, that determines the amplitudes of inhomogeneous *in-phase* oscillations of the assembly (1). But due to the symmetry of system (3) only in $N - 2$ of these states the assembly (1) has genuinely distinct behaviour. Here elements form two clusters: n of them have "high" amplitude $p_{\rm A}$ and $(N - n)$ have "low" amplitude $q_{\rm A}$, where $n = 1, 2, ..., N - 1$.

For example, Figure 4b shows amplitude clusters (solid circles), which have formed from the initial distribution marked by open ones in Figure 4a. Note that this property of equal amplitude in the clusters in a system with global coupling is *exact*, and not approximate as in systems with diffusive coupling [31]. The stability conditions of amplitude clusters depend not only on the parameters



(a)



(b)

Fig. 4. One of the stable in-phase states ($N = 30, a = 18, \beta = 0.2$). (a) Initial state, (b) final distribution of complex amplitudes (14 elements have "low" amplitude and 16 "high" amplitude).

of system (3), but also on the values of N and n (to be precise on their ratio). Therefore, roughly speaking, solutions with different n (for fixed N and a) loose stability for different values of β. The stability boundary and regions of existence for amplitude clusters with different n for the assembly with $N = 30$ are shown in Figure 5. Numerical exploration shows that lines of stability and existence are very close and for each fixed a have a maximum whose location depends on the value of a. Hence clusters having marked predominance of elements with high or low amplitude loose stability first.

Thus when the parameters change within the region $d_{\rm cl}^{-}$ from the curve B_1 to B_2 in Figure 3, amplitude clusters loose stability one at a time. And when B_2 is reached, all of them become unstable. For the value of the parameters in the region above B_2 only homogeneous oscillations are stable.

4 Amplitude-phase clusters

Consider now the collective behaviour of the assembly (1), when all oscillators are *nonisochronous* ($\alpha \neq 0$). We take here the coefficient of coupling between the oscillators complex ($\beta, \gamma \neq 0$). In this case from the equations for the

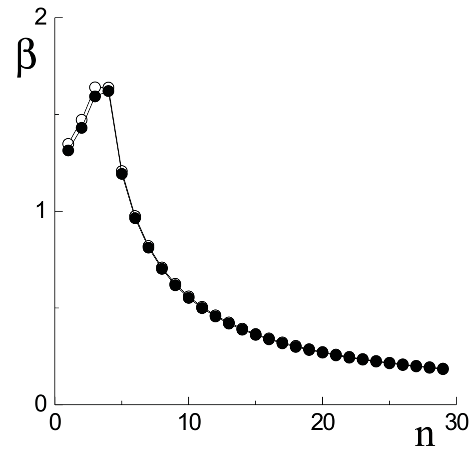


Fig. 5. Boundaries of existence (open circles) and stability (solid circles) of amplitude clusters with different n for $N = 30$, $a = 18$.

phases, φ_j $\left(z_j = r_j \mathrm{e}^{i\varphi_j}\right)$, follows that in-phase motions are impossible in the system (3).

Let us look for the solution of the system (3) in the form

$$r_j(t) = \begin{cases} p(t), & j = 1, 2, \dots, n, \\ q(t), & j = n+1, \dots, N \end{cases} \tag{24}$$

and

$$\varphi_j(t) = \begin{cases} \psi_1(t), & j = 1, 2, \dots, n, \\ \psi_2(t), & j = n+1, \dots, N. \end{cases} \tag{25}$$

From (3) follows the evolution equations for amplitudes, $p(t), q(t)$, and the phase difference, $\psi = \psi_2 - \psi_1$:

$$\begin{cases} \dot{p} = -Q(p) + q\dfrac{N-n}{N}[\beta\cos\psi - \gamma\sin\psi], \\[2ex] \dot{q} = -R(q) + p\dfrac{n}{N}[\beta\cos\psi + \gamma\sin\psi], \\[2ex] \dot{\psi} = \alpha\left[g(q) - g(p)\right] + \gamma\cos\psi\left[\dfrac{np}{Nq} - \dfrac{(N-n)q}{Np}\right] \\[2ex] \qquad -\beta\sin\psi\left[\dfrac{np}{Nq} + \dfrac{(N-n)q}{Np}\right] + \dfrac{N-2n}{N}\gamma, \end{cases} \tag{26}$$

with

$$Q(p) \equiv p\left[f(p) + \beta\frac{N-n}{N}\right],\; R(q) \equiv p\left[f(p) + \beta\frac{N-n}{N}\right].$$

Let us assume that β (26) satisfies the condition:

$$\beta < \begin{cases} \dfrac{(a-8)N}{8(N-n)}, & n \leq N/2, \\[2ex] \dfrac{(a-8)N}{8n}, & n > N/2. \end{cases} \tag{27}$$

Then the functions $Q(p)$ and $R(q)$ satisfy the conditions:

$$\begin{array}{ll} Q(p_i)=0,\ R(q_i)=0,\ i=0,1,2, & \\ Q(p)>0, & \text{if } p\in(0,p_1) \text{ and } p>p_2, \\ Q(p)<0, & \text{if } p\in(p_1,p_2), \\ R(q)>0, & \text{if } q\in(0,q_1) \text{ and } q>q_2, \\ R(q)<0, & \text{if } q\in(q_1,q_2), \end{array} \tag{28}$$

where

$$p_0=0,\quad p_{1,2}=\frac{1}{2}\sqrt{1\mp\sqrt{1-\frac{8}{a}\left(1+\beta\frac{(N-n)}{N}\right)}},$$
$$q_0=0,\quad q_{1,2}=\frac{1}{2}\sqrt{1\mp\sqrt{1-\frac{8}{a}\left(1+\beta\frac{n}{N}\right)}}.$$

Let us show that there exists an invariant domain in the phase space of the system (26). Consider the following region

$$\Omega=\{\ p,q: p_2-A\le p\le p_2+A,\ 0\le q\le B\},$$

where A and B are the parameters satisfying the conditions

$$0<A<P_2-p_1,\ 0<B<p_1. \tag{29}$$

The boundary of the region Ω is a cylindrical surface. Consider the orientation of the vector field at this surface.

Let us take $q>0$. In this case from the system (26) we obtain

$$\begin{aligned} \dot{p}_{|p=p_2+A} &\le -Q(p_2+A)+q\frac{N-n}{N}\sqrt{\beta^2+\gamma^2}, \\ \dot{p}_{|p=p_2-A} &\ge -Q(p_2-A)-q\frac{N-n}{N}\sqrt{\beta^2+\gamma^2}, \\ \dot{q}_{|q=B} &\le -R(B)+p\frac{n}{N}\sqrt{\beta^2+\gamma^2}. \end{aligned} \tag{30}$$

Let us find the conditions when, for $q>0$, the vector field of (26) at the boundary of Ω is oriented into this region. It happens, when the parameters A and B satisfy the conditions (29) and the following inequalities

$$A\le -p_2+\frac{NR(B)}{n\sqrt{\beta^2+\gamma^2}},\ B\le -\frac{NQ(p_2-A)}{(N-n)\sqrt{\beta^2+\gamma^2}}. \tag{31}$$

If the system (31) has a solution, the existence of the region D in the plane $(A,\ B)$ is ensured (Fig. 6). Note that not all parameter values allow solution for the system (31). Let us mark by d_r the set of the parameter values of the system (26), when there exists the region D. For example, this region always exists if the following condition is fulfilled:

$$R(q_{\max})\ge\frac{n}{N}\sqrt{\beta^2+\gamma^2}\,(2p_2-p_{\max}), \tag{32}$$

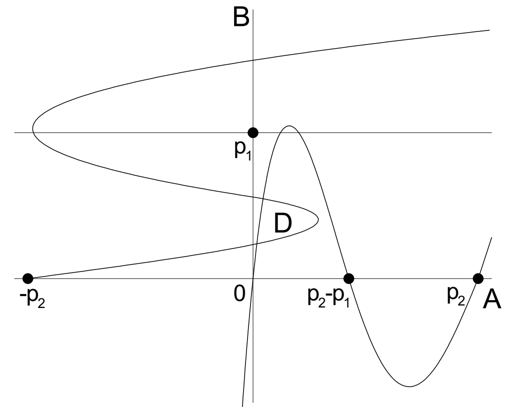


Fig. 6. Graphic solution of the inequalities (31) for $N=30$, $n=4$, $a=20$, $\beta=1$, $\gamma=0.1$.

where

$$q_{\max}=\sqrt{\frac{3}{20}-\frac{1}{10}\sqrt{\frac{9}{4}-\frac{10}{a}\left(1+\beta\frac{n}{N}\right)}},$$
$$p_{\max}=\sqrt{\frac{3}{20}+\frac{1}{10}\sqrt{\frac{9}{4}-\frac{10}{a}\left(1+\beta\frac{N-n}{N}\right)}},$$

hence the inequality (32) gives only an approximate location for the boundary of the region d_r.

Consider the orientation of the vector field at the surface $\{q=0\}$. It follows from the system (26), that for $q=0$ the phase ψ_2 can have an arbitrary value. However, it must satisfy the equation

$$\gamma\cos(\psi_2-\psi_1)-\beta\sin(\psi_2-\psi_1)=0. \tag{33}$$

Substituting (33) in (26) we obtain that in the plane $\{q=0\}$ the vector field obeys the equation

$$\dot{q}_{|q=0}=p\frac{n(\beta^2+\gamma^2)}{N\beta}\cos(\psi_2-\psi_1). \tag{34}$$

We can determine the phase ψ_2 such that

$$\dot{q}_{|q=0}>0. \tag{35}$$

Thus for the parameter values taken from the region d_r the vector field of (26) is oriented inwards to the region Ω. A qualitative sketch of the intersection of the region Ω with the plane $\{\psi=\text{const}\}$ and the orientation of the vector field of (26) at the boundary of the region are given in Figure 7. Let us mark by Ω^+ the part of the region Ω between the planes $\{\psi=0\}$ and $\{\psi=\pi-\arctan\gamma/\beta\}$. The orientation of the vector field on these planes can be

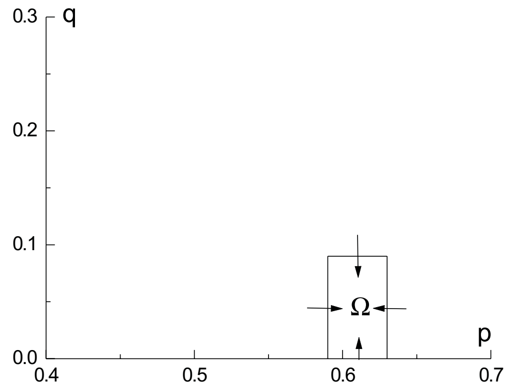


Fig. 7. Orientation of vector field at the boundary of the region Ω.

found from (26):

$$\begin{cases} \dot{\psi}_{|\psi=0} = \alpha\left[g(q) - g(p)\right] + \dfrac{N-2n}{N}\gamma \\ \qquad +\gamma\left[\dfrac{np}{Nq} - \dfrac{(N-n)q}{Np}\right], \\ \dot{\psi}_{|\psi=\pi-\arctan\gamma/\beta} = \alpha\left[g(q) - g(p)\right] \\ \qquad +\dfrac{N-2n}{N}\gamma - \dfrac{2\beta\gamma}{\sqrt{\beta^2+\gamma^2}}\dfrac{np}{Nq}. \end{cases} \tag{36}$$

On the other hand, the trajectories of the system (26) belonging to the region Ω, satisfy the inequalities

$$\begin{array}{c} p_2 - A_0 < p(t) < p_2 + A_0, \\ 0 < q(t) < B_0. \end{array} \tag{37}$$

Using (37), we obtain from (36), that if the parameters of the system (26) belong the region d_ψ, given by

$$\begin{aligned} &|\alpha|G_{\max} + \frac{|N-2n|}{N}\gamma - \frac{n\beta\gamma}{\sqrt{\beta^2+\gamma^2}}\frac{p_2 - A_0}{B_0} < 0, \\ &|\alpha|G_{\max} - \frac{|N-2n|}{N}\gamma - \gamma\left[\frac{n(p_2-A_0)}{B_0} - \frac{(N-n)B_0}{p_2 - A_0}\right] < 0, \end{aligned} \tag{38}$$

with

$$G_{\max} \equiv g(B_0) + \max\left\{g(p_2 - A_0), -g(p_2 + A_0)\right\}$$

the vector field is oriented inwards to the region Ω^+. Thus, all along the surface of the region Ω^+ the vector field of the system (26) points inwards to this region, and hence Ω^+ is the invariant domain.

Due to invariant property of the region Ω^+ in the phase space of the system there exists at least one trajectory L, satisfying the condition

$$\big(p(t), q(t), \psi(t)\big) \in \Omega^+. \tag{39}$$

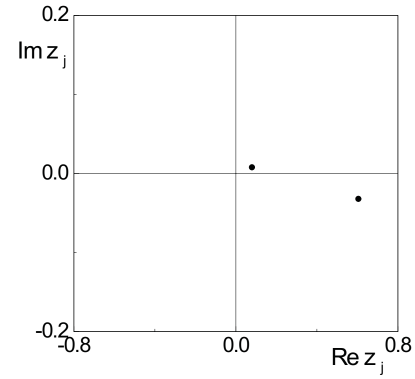
It can be shown, that the components $p(t), q(t)$ (the amplitudes of oscillations of the system (3)) satisfy the inequalities (37), and the phase differences satisfy the following inequality:

$$0 < \psi_2(t) - \psi_1(t) < \pi - \arctan\gamma/\beta.$$

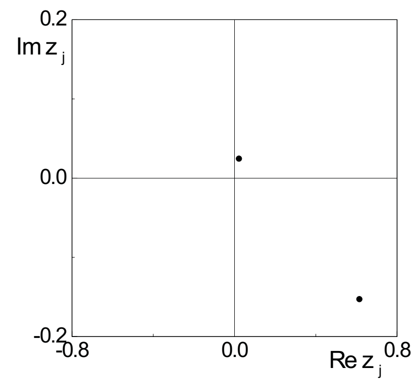
Thus, the trajectory L determines the oscillatory behaviour of the system (3) in the form (24, 25), *i.e.* in the form of amplitude-phase clusters. Such clusters exist for the values of the parameters from the region $\{d_r \bigcap d_\psi\}$.

Numerical integration of the evolution equations (3) permits verification of the conditions obtained above. The amplitude-phase clusters exist in a rather "wide" range of parameter values. Figure 8 shows two typical solutions in the form of the amplitude-phase clusters. The initial state is an almost in-phase state with a random distribution of amplitudes. Integration of the system (3) shows not only the existence of the amplitude-phase clusters, but also their stability.

Note, that the existence of the amplitude-phase states is impossible in assemblies composed of oscillators with a single attractor (*e.g.* limit-cycle oscillators).



(a)



(b)

Fig. 8. Amplitude-phase clusters. Parameter values $a = 20$, $N = 30$ (26 with "low" amplitude and 4 with "high" amplitude) (a) $\alpha = -0.1$, $\beta = 1$, $\gamma = 0.1$, (b) $\alpha = -10$, $\beta = 0.5$, $\gamma = 0.2$.

5 "Splay-phase" states

The collective state of an assembly when the amplitudes of oscillations are the same but the phases are different and such that $\varphi_j(t) = \Phi_j$, where Φ_j are constants, satisfying the condition $\sum_{j=1}^{N} \mathrm{e}^{i\Phi_k} = 0$, is called a "splay-phase" state [32,33]. Let us show, that the assembly (1) has stable "splay-phase" states in another albeit related way.

Taking into account the condition $\sum_{j=1}^{N} \mathrm{e}^{i\Phi_k} = 0$, we find from (3) the following solution:

$$z_j(t) = \begin{cases} r^{(3)} \mathrm{e}^{i\left(g(r^{(3)})t+\Phi_j\right)}, & j = 1, 2, \ldots, n, \\ 0, & j = n+1, \ldots, N, \end{cases} \tag{40}$$

with

$$r = \frac{1}{2}\sqrt{1 + \sqrt{1 - 8(1+\beta)/a}} \quad \text{and} \quad \sum_{j=1}^{n} \mathrm{e}^{i\Phi_j} = 0.$$

The solution (40) exists in the region of parameter values defined by the inequality $\beta < a/8 - 1$. Besides, for system (3) there is a splay-phase state with amplitude $r = r^{(4)}$, where $r^{(4)} \equiv \frac{1}{2}\sqrt{1 - \sqrt{1 - 8(1+\beta)/a}}$. But, as shown below, such a state is linearly (locally) unstable hence we disregard it. Note also that in (40) the index n is arbitrary and, in particular, can be equal to N. In the later case, (40) defines a splay-phase state typical of assemblies of globally coupled limit-cycle oscillators as discussed in reference [16].

Let us now consider the stability of the splay-phase states (40). Linearizing the system (3) around the corresponding solution, we obtain for perturbations, $\xi_j \in \mathrm{C}$, the following system:

$$\begin{cases} \dot{\xi}_j = \dfrac{\beta + i\gamma}{N} \displaystyle\sum_{k=1}^{N} \xi_k \mathrm{e}^{i(\Phi_k - \Phi_j)} \\ \quad - (a + i\alpha) H(r^{(3)})(\xi_j + \xi_j^*), \quad j = 1, 2, ..., n, \\ \dot{\xi}_j = \dfrac{\beta + i\gamma}{N} \displaystyle\sum_{k=1}^{N} \xi_k \mathrm{e}^{i(\Phi_k - \Phi_j)} - (\beta + 1)\xi_j, \\ j = n+1, ..., N. \end{cases} \tag{41}$$

To the system (41) we associate a $2N \times 2N$ matrix whose eigenvalues are the Lyapunov exponents of the solution (40). Upon transformation of the matrix to the block-diagonal form follows that all the eigenvalues split in two groups. The first group contains $N - n - 4$ negative eigenvalues equal to $-(\beta + 1)$ and $n - 4$ zero eigenvalues corresponding to the dimension of the manifold of locked fixed phases. The second group consists of eight eigenvalues, two of which are always negative and equal to $-(\beta + 1)$, and the rest of them are the roots of the characteristic equation

$$P(\lambda)P^*(\lambda) - (a^2 + \alpha^2)(\beta^2 + \gamma^2)H^2(r^{(3)}) \\ \times \Delta^2(\lambda + \beta + 1)^2 = 0, \tag{42}$$

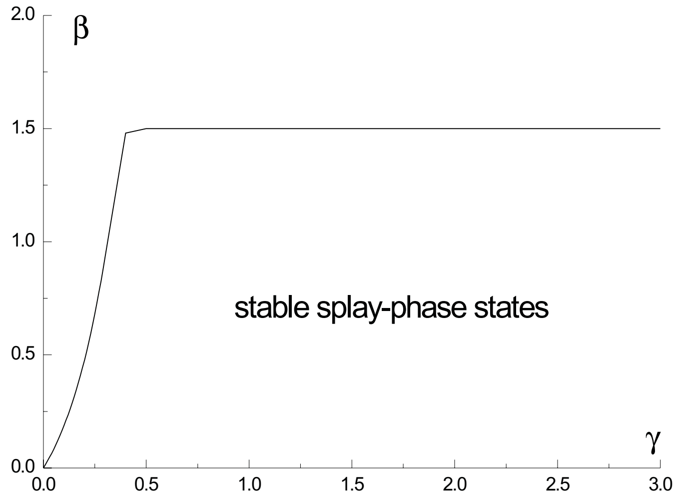


Fig. 9. Stability region of splay-phase states for $a = 20$, $\alpha = -30$.

with

$$\begin{aligned} P(\lambda) = {} & \lambda^3 + \lambda^2(2aH(r^{(3)}) + 1 - i\gamma) \\ & + \lambda\Big\{2a(\beta+1)H(r^{(3)}) - (\beta + i\gamma) \\ & \times \left[\left(2a - \frac{n(a + i\alpha)}{N}\right) H(r^{(3)}) + \frac{(\beta+1)n}{N}\right]\Big\} \\ & - \frac{(\beta+1)n}{N}(a\beta + \alpha\gamma + i(a\gamma - \alpha\beta))H(r^{(3)}), \end{aligned}$$

$$\Delta = \left| \frac{1}{N} \sum_{k=1}^{N} \mathrm{e}^{2i\Phi_k} \right|.$$

Hence, the stability conditions of (40) depend essentially on the parameter Δ, *i.e.* on the distribution of phase constants, Φ_j. For $\Delta = 0$ the stability boundary for splay-states of the form (40) is given in Figure 9. For the parameter values from the region below the line there can exist stable splay-states of two types. The first type is illustrated in Figures 10a, b and the second in Figures 10c, d. In the first case $(N - n)$ oscillators are at rest and the other n are periodically oscillating. When the second type appears all the oscillators are excited and indeed oscillate. Note that splay-phase states of the first type occur due to the *bistability* of the unit and do not exist in assemblies of oscillators with a single limit cycle. The solution of the second type is not related to the bistable properties of the units as shown in [16,25].

6 Collective chaos

Numerical integration of the system (3) shows that for some parameter values the collective dynamics of the assembly (1) can be chaotic (there is a positive Lyapunov exponent). It can be shown that the salient dynamical

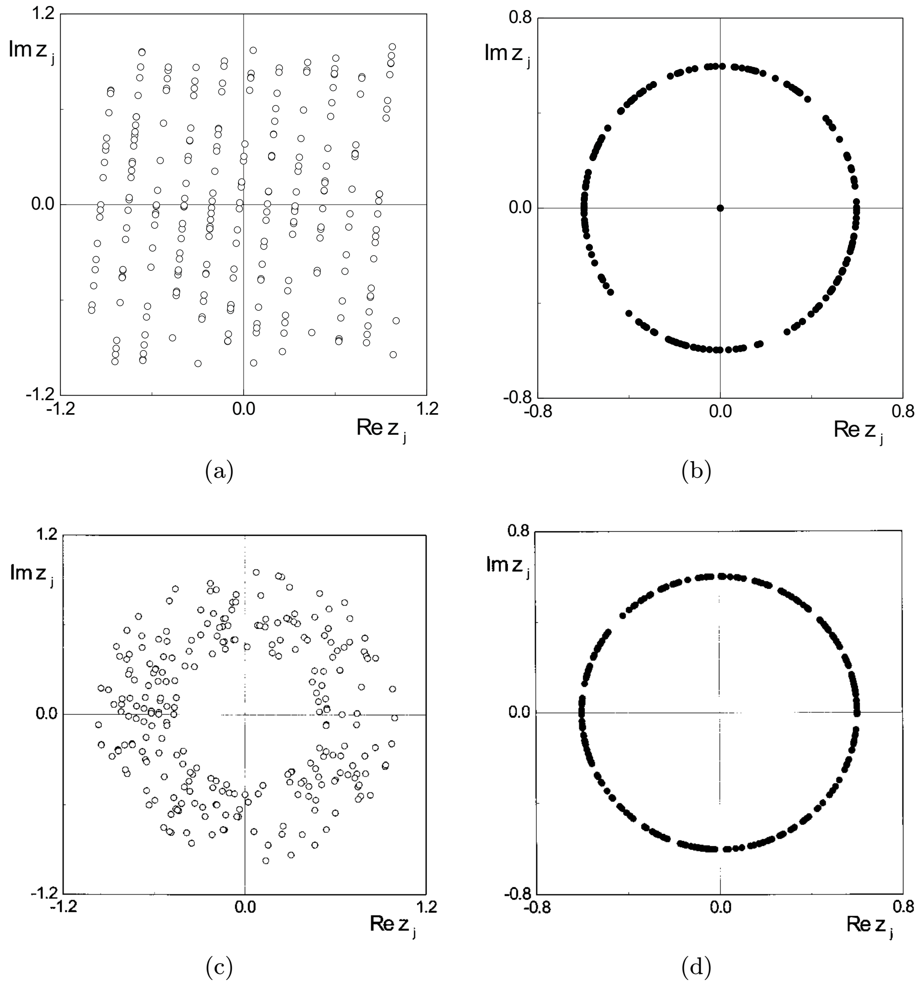


Fig. 10. Two types of splay-phase states for $a = 20$, $\alpha = -30$, $\beta = 1$, $\gamma = 2$, $N = 256$: (a) initial state, (b) the first type (39 elements have zero amplitude and 217 high amplitude); (c) initial state, (d) the second type (final distribution for initial conditions in Fig. 10c).

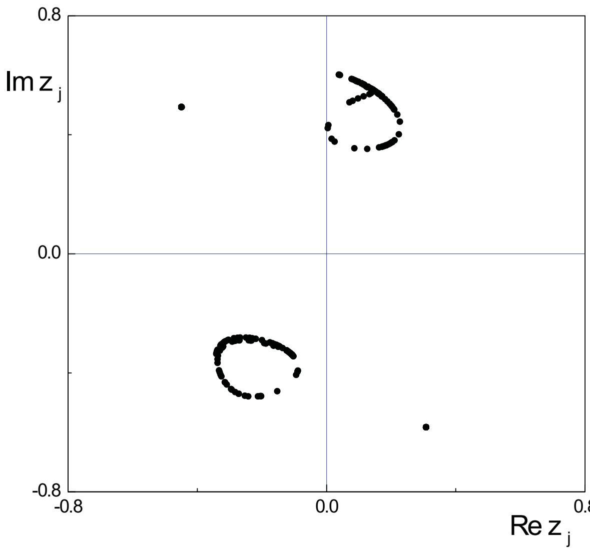
properties of this regime are the same as for the "ρ-shaped type" regime in the case of globally-coupled limit cycle oscillators as discussed in reference [16]. The form of the ρ-shaped distribution of the oscillators of the assembly (1) is plotted in Figure 11a. Each oscillator in this regime is forced by the mean field, which itself is chaotic. The evolution of the order parameter, $M = |\bar{z}|$, is drawn in Figure 11b. Individual oscillators move around the ρ-shaped loop with eventual jumps to the tail part. In the assembly (1) this regime has some new features. When the strength of the coupling becomes large enough (Fig. 11a), a number of oscillators separates from the tail part. These oscillators form a cluster (single dot in Fig. 11a) located at some distance from the ρ-shaped loop. Their motion is periodic and regular while the dynamics of oscillators in the ρ-shaped loop is chaotic (Fig. 12).

It also appears that the system (1) possesses the following property of multistability. Depending on the initial conditions for the same parameter values one of the following regimes can be attained: (i) all the oscillators are at rest; (ii) the assembly exhibits collective chaos; and (iii) oscillators form a splay-phase state.

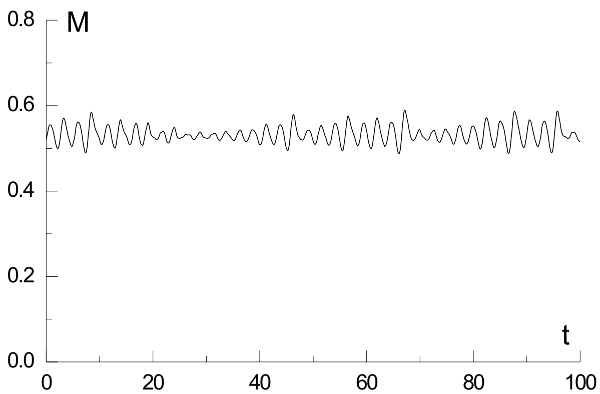
Conclusion

The model studied here is of potential interest to understand the dynamics of large neural oscillatory networks or lattices [3,7–9]. Such systems are characterized by rather complex intralattice connections. Our global coupling may prove valuable to mimick real behavior. Furthermore, the bistability of units captures an ingredient of the bistability of neurons with coexistence of the state of rest and excited states. We have shown that this property of units leads to interesting new features in the collective behavior of the globally coupled network:

(i) Amplitude-phase clusters can form in the system (1). Then all units or, say, neurons (here mimicked by bistable oscillators) break into two groups. The first



(a)



(b)

Fig. 11. Collective chaos in the assembly for $a = 20$, $\alpha = -30$, $\beta = \gamma = 1.8$: (a) Snapshots of the 256 oscillators (dots) in the complex plane at two different instants of time, (b) Evolution of the order parameter M.

group consists of "strongly" excited neurons having rather "high" oscillation amplitude, hence *spiking*. The second group is composed of "weakly" excited neurons with a "low" oscillation amplitude, hence *subthreshold* oscillations. Furthermore, neurons taken from different groups oscillate with a constant phase shift (see *e.g.* [8]).

(ii) The system (1) can operate in a mode such that part of the neurons exhibits chaotic oscillations while the other part oscillates regularly, hence regular and chaotic oscillations form a "linked" state.

(iii) The system (1) can have two types of multistability. In the first case, collective chaos and regular dynamics (splay-phase states and the trivial state) "compete" with each other. In the second case, the competition occurs between the amplitude-phase clusters, splay-phase states and the trivial state. Note that although a "winner" in this competition is one of the large number of such splay-phase states, there is a difference in the behavior of the assembly (1) in the two cases. In the first case, part of the neurons (here mimicked by bistable oscillators) is excited (splay-phase state) and the other part is at rest. In the second case, *all* neurons are either excited or at rest.

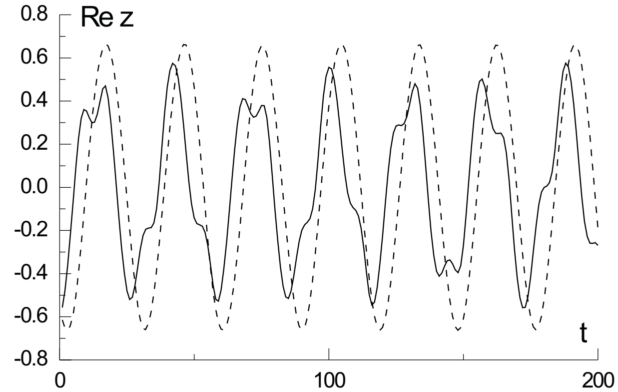


Fig. 12. Time evolution of two elements with regular and chaotic motion within the chaotic attractor for the parameter values of Figure 11. The real parts of oscillation amplitudes are shown. The broken line shows the regular motion (element with $j = 1$), and the solid line shows the chaotic motion ($j = 5$).

The authors acknowledge fruitful discussions with Prof. R. Llinas, Prof. A. Fdez. de Molina, Prof. L.O. Chua, Dr. V. Hakim, Dr. J. Welsh, Dr. E. Lang, and Ms. M.-L. Chabanol. This research has been supported by the BCH Foundation, by the Russian Foundation for Basic Research under Grant 97-02-16550, by DGICYT under Grant PB 96–599 and by NATO under Grant OUTR LG 96–578. V.I.N. would like to thank the Ministry of Education and Culture (Spain) for economic support.

References

1. H. Haken, *Advanced Synergetics* (Springer, Berlin 1983).
2. H. Haken, *Information and Self–Organization* (Springer, Berlin 1988).
3. H. Haken, *Principles of Brain Functioning* (Springer, Berlin 1996).
4. A.T. Winfree, *The Geometry of Biological Time* (Springer, New York 1980).
5. Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin 1984).
6. V.S. Afraimovich, V.I. Nekorkin, G.V. Osipov, V.D. Shalfeev, *Stability, Structures and Chaos in Nonlinear Synchronization Networks* (World Scientific, Singapore 1995).
7. See e.g. R. Llinas (Ed.) *The Biology of the Brain. From Neurons to Networks* (W.H. Freeman, N.Y. 1988).
8. R. Llinas, Y. Yarom, J. Physiol. **376**, 163 (1986).
9. B. Delord, A.J. Klaassen, Y. Burnod, R. Costalat, E. Guigon, Neuroreport **8**, 1019 (1997).
10. S.V. Chakravarthy, J. Ghosh, Biol. Cybern. **75**, 229 (1996).

11. G.N. Borisyuk, *et al.*, Math. Modeling **4**, 3 (1992) (in Russian).
12. P. Hadley, M.R. Besley, K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988).
13. S. Nichols, K. Wiesenfeld, Phys. Rev. E **50**, 205 (1994).
14. Y. Nishio, A. Ushida, IEEE Trans. Circuits Syst. I: Fundamental Theory Appl. **42**, 678 (1995).
15. D.G. Aronson, Y.S. Huang, Nonlinearity **7**, 777 (1994).
16. V. Hakim, W.–J. Rappel, Phys. Rev. A **46**, 7347 (1992).
17. M.Y. Choi, Y.W. Kim, D.C. Hong, Phys. Rev. E **49**, 3825 (1994).
18. J.-A. Sepulchre, R.S. MacKay, *Proc. of the 3 Int. Spec. workshop on Nonlinear Dynamics of Electronic Syst. – NDES'95*, Dublin, Ireland (1995), p. 295.
19. J.F. Heagy, T.L. Carroll, L.M. Pecora, Phys. Rev. A **50**, 1874 (1994).
20. N. Nakagawa, Y. Kuramoto, Physica D **80**, 307 (1995).
21. K. Okuda, Physica D **63**, 424 (1993).
22. K.Y. Tsang, R.E. Mirollo, S.H. Strogatz, K. Wiesenfeld, Physica D **48**, 102 (1991).
23. H. Sakaguchi, S. Shinomoto, Y. Kuramoto, Prog. Theor. Phys. **77**, 1005 (1987); *ibidem* **79**, 1069 (1988); see also S.H. Strogatz, R. Mirollo, Physica D **31**, 143 (1988); J. Phys. A **21**, L699 (1988); H. Daido, Phys. Rev. Lett. **61** (1998) 231.
24. D. Golomb, D. Hansel, B. Shraiman, H. Sompolinsky, Phys. Rev. A **45**, 3516 (1992). See also D. Hansel, G. Mato, C. Meunier, Phys. Rev. E **48**, 3470 (1993); P. Tass, Phys. Rev. E **56**, 2043 (1997); T.W. Carr, I.B. Schwartz, Physica D **115**, 321 (1998).
25. P.C. Matthews, R.E. Mirollo, S.H. Strogatz, Physica D **52**, 293 (1991).
26. K. Kaneko, Physica D **41**, 137 (1990).
27. N. Nakagawa, Y. Kuramoto, Physica D **75**, 74 (1994).
28. T. Mizuguchi, M. Sano, Phys. Rev. Lett. **75**, 966 (1995).
29. H. Daido, Phys. Rev. Lett. **77**, 1406 (1996).
30. R. Bellman, *Introduction to Matrix Analysis*, 2nd ed., (McGraw–Hill, New York 1970).
31. V.I. Nekorkin, V.A. Makarov, Phys. Rev. Lett. **74**, 4819 (1995).
32. K. Wiesenfeld, P. Hadley, Phys. Rev. Lett. **62**, 1335 (1989).
33. J.W. Swift, S.H. Strogatz, K. Wiesenfeld, Physica D **55**, 239 (1992).
34. S. Nichols, K. Wiesenfeld, Phys. Rev. A **45**, 8430 (1992).